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Note

On Computing Electrostatic Field Lines for Two-Dimensional Vacuum Fields in the Neighborhood of Localized Regions of Charge

In the course of scientific research, the clear presentation of results can be just as important as the results themselves. This is certainly true with computational physics which is characterized by large codes producing many numbers. For example, electric fields are usually depicted by vector plots or contours of equal potentials. The potential function is relatively easy to compute, and its level curves are easy to plot with most graphics packages. Frequently, however, the electric field lines give a clearer description of the physics of the system.

For many systems, equations for the field lines can be derived analytically [1, 2] and the results can be plotted directly [3]. However, if an analytic solution does not exist, a computational one must be found. Whittaker developed one technique based on plotting trajectories in fields [4]. His method is the basis for the STRMLN field line plotting routine in the graphics package supported by the National Center for Atmospheric Research [5]. It has a major drawback in that the density of field lines does not indicate the strength of the fields. Here, we present a method for obtaining electric field lines such that the density of field lines is directly related to the strength of the field. To do this, a function is found with level curves coincident with the electric field. The level curves of this function can then be plotted with any standard contour plotting routine.

Given the potential ϕ of an electrostatic system varying in only two dimensions $(x-y)$, we can construct field lines. To do this, we find another function ψ, the level curves of which: (1) are everywhere perpendicular to the level curves of ϕ; and (2) lie in the $(x-y)$ plane. Thus the level curves of ψ are the field lines. Equivalently, we find ψ such that the gradient of ψ is everywhere perpendicular to the gradient of ϕ, and the gradient lies in the $(x-y)$ plane. Thus,

$$\nabla\psi = \nabla\phi \times \mathbf{z}. \tag{1}$$

Note that Eq. (1) implies that ϕ and ψ satisfy the Cauchy–Riemann equations. From this point of view, it can also be shown that ϕ and ψ are orthogonal and therefore, the contours of constant ψ (commonly called the stream function) represents the field lines [1, 2].

Since $\mathbf{E} = -\nabla\phi$ the magnitude of the electric field can be deduced from the den-

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sity of lines on a contour plot of ϕ if these contours represent regular increments in ϕ. The same can be said for ψ. From Eq. (1),

$$-\nabla\psi = \mathbf{E}\times\mathbf{z} \Rightarrow \nabla\psi\cdot\nabla\psi = (\mathbf{E}\times\mathbf{z})\cdot(\mathbf{E}\times\mathbf{z})$$
$$= (\mathbf{E}\cdot\mathbf{E})\cdot(\mathbf{z}\cdot\mathbf{z}) - 2(\mathbf{E}\cdot\mathbf{z}) = E^2;$$

so again, the density of field lines indicates the strength of the field with the field directed *along* the lines rather than normal to them.

We have $\nabla\psi$, but to find ψ we must integrate. For any path, s in the $x-y$ plane

$$\frac{\partial\psi}{\partial s} = \nabla\psi\cdot\mathbf{s},$$

where $\mathbf{s}$ is the unit vector along the path at every point. From Eq. (1)

$$\frac{\partial\psi}{\partial s} = (\nabla\phi\times\mathbf{z})\cdot\mathbf{s} = \nabla\phi\cdot(\mathbf{z}\times\mathbf{s}).$$

Note that $\mathbf{z}\times\mathbf{s} = -\mathbf{n}$, where $\mathbf{n}$ is normal to curve s pointing towards the right while traversing s. Thus,

$$\frac{\partial\psi}{\partial s} = -\nabla\phi\cdot\mathbf{n} \qquad \text{or} \qquad \psi(s_2)-\psi(s_1) = -\int_{s_1}^{s_2}\nabla\phi\cdot\mathbf{n}\,ds. \tag{2}$$

Clearly, ψ, like ϕ, is arbitrary to within an additive constant.

The integral of $\partial\psi/\partial s$ around any closed path, s, is zero if the path does not enclose a net charge. However, this is not so when the path does enclose a net charge. From Gauss' law (modified to two dimensions assuming no variation in $\mathbf{z}$), the integral around a closed path is

$$\oint\nabla\phi\cdot\mathbf{n}\,ds = 4\pi\lambda,$$

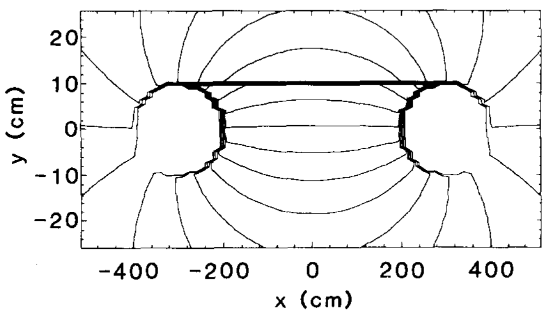

FIG. 1. Field lines for the two cylinder case. The lines were computed by integrating Eq. (2) with no regard for the branch cuts.

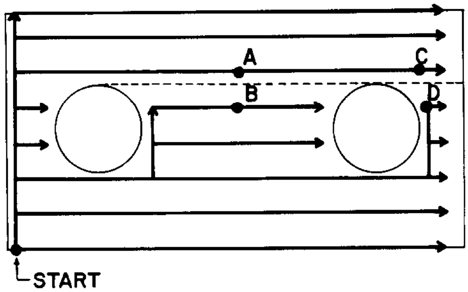

FIG. 2. The path of integration used for the field line computations of Fig. 1.

where λ is the line charge enclosed in the loop. Thus ψ is not a single-valued function in a domain enclosing localized regions of charge. To calculate ψ from Eq. (2) we will be forced to integrate around the charges; ψ will be discontinuous along the line where these paths of integration meet. This line of discontinuity is called a branch cut.

Figure 1 shows an example of a branch cut due to computing ψ by simple integration. In this case we have two infinite, conducting cylinders with opposite line charge $(\pm\lambda)$ in free space. The potential for this case is given by

$$\phi(x, y) = \lambda \ln \frac{(x+d)^2 + y^2}{(x-d)^2 + y^2},$$

where $d^2 = r^2 - a^2$, r being half the distance between the cylinders and a being the radius of the cylinders.

The integration started in the lower left corner and proceeded as shown in Fig. 2. The paths of integration meet in two places as shown by the dotted lines. There is a discontinuity in ψ along the first line but ψ is continuous along the second. This is because the loop around the left conductor encloses a net charge while the loop around both conductors encloses no net charge.

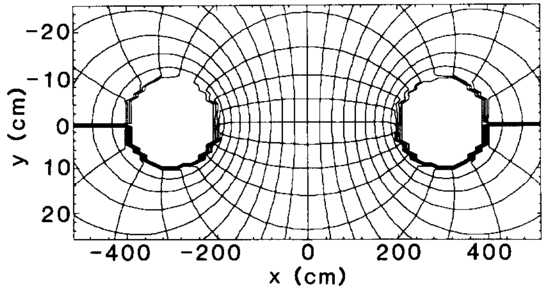

FIG. 3. Field lines using alternate integration path. The equipotential contours have been superimposed to show orthogonality.

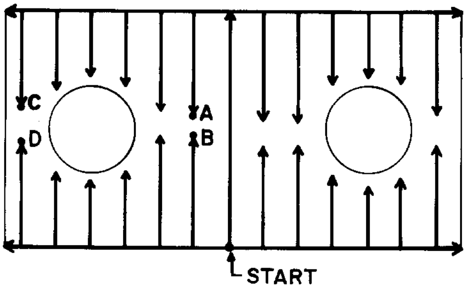

FIG. 4. Alternate integration path.

To make a nicer picture, one simply needs to move the branch cuts to different locations. Figure 3 shows the result of a different path of integration. The potential contours have been superimposed to show that ψ is indeed orthogonal to ϕ. For this case, the path of integration was chosen judiciously to place the branch cuts along a line of symmetry and to remove them from the center of the figure to make them much less noticeable (Fig. 4).

More complicated situations can be solved by more complicated paths. Figure 5 shows the field lines for a quadrupole system. In this case, the boundary is a conducting wall and the potential was calculated by another program.

To remove all branch cuts from the center of the figure, the integration was done in a spiral pattern (Fig. 6). The prescription is to start in the center with $\psi = \psi_0$ (arbitrarily set to 0), then traverse the spiral. At each point, compute ψ by

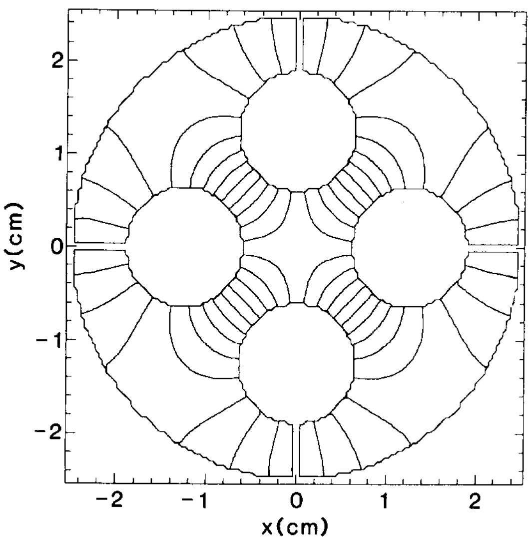

FIG. 5. Quadrupole system in a conducting cylinder. Here also, the branch cuts have been moved to the outside of the system.

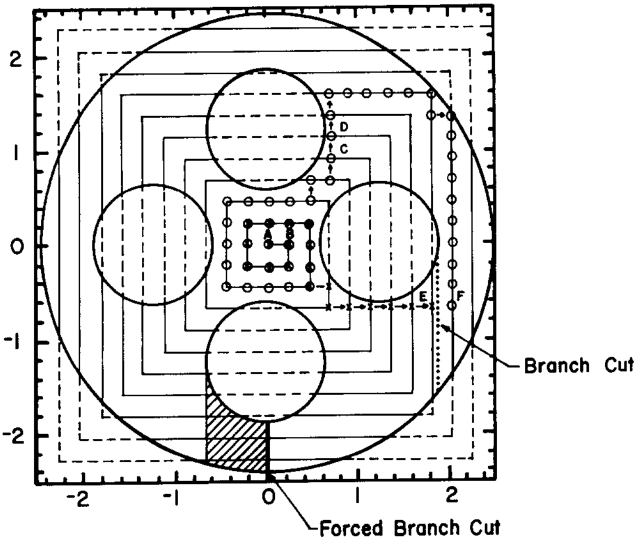

FIG. 6. Integration path for quadrupole system. The computational grid was traversed in a spiral. However, since the integration can not pass through the conductors, the direction of integration is not always along the spiral path. This will produce branch cuts, one of which is shown by the dotted lines. These cuts can be moved by restricting the integration.

integrating along the spiral from the previous point. For example, ψ at B is computed by integrating Eq. (2) from A to B.

Inevitably, the spiral path will cross conductors and regions exterior to the system. In these cases, the value of ψ at the first point which reenters the system is calculated from another previously computed grid point. For example, ψ at D is calculated by integrating from point C.

The result is that even though the grid is traversed in a spiral pattern, the path to a particular point will not necessarily be along the spiral. The full paths for two points, E and F, are shown in Fig. 6. These paths surround a conductor so they are on opposite sides of a branch cut. The branch cut is shown as a dotted line.

The branch cuts are moved to more aesthetic locations by not allowing the integration to pass predetermined branch cuts, one of which is shown. This forces the position of the branch cut but leaves ψ not computed in certain regions. One such region is shown by the hashed area. These points are computed by integrating backwards after the spiral is completed, giving the final result shown in Fig. 5.

These techniques can be used to calculate and plot field lines in many complex geometries which include localized regions of charge. Enveloped regions of charge necessarily cause branch cuts in the values of ψ but these branch cuts can be moved by changing the paths of integration.

References

1. W. R. Smythe, *Static and Dynamic Electric Fields*, 3rd ed. (McGraw-Hill, New York, 1968), p. 74.
2. E. Butkov, *Mathematical Physics* (Addison-Wesley, Reading, MA, 1968), p. 99.
3. R. H. Kirchhoft, *Potential Flows, Computer Graphic Solutions* (Dekker, New York, 1985).
4. T. M. Whittaker, *Monthly Weather Rev.* **105**, 786 (1977).
5. F. Clare, D. Kennison, and B. Lackman, *NCAR Graphics User's Guide*, Version 2.00 (National Center for Atmospheric Research, Boulder, 1987), p. 340.

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